Responsiveness and Manipulability of Formation of Multi-Robot Networks

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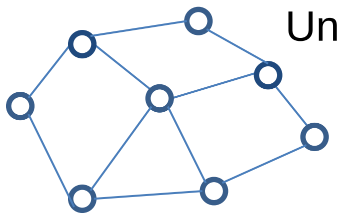
Distance-Based Formation Control

- Interaction rule for follower agent *i* :



Underlying graph $G = (V, E)$

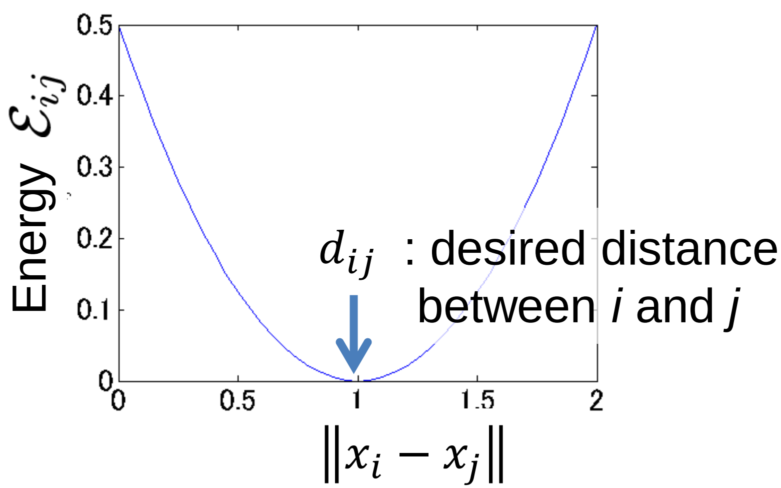
$x_i \in \mathbb{R}^d$: state (position)

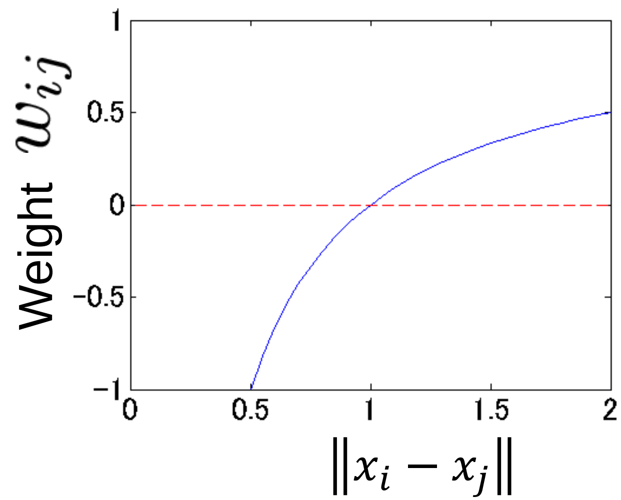
$$\dot{x}_i(t) = -\sum_{j \in \mathcal{N}(i)} \frac{\partial \mathcal{E}_{ij}(\|x_i - x_j\|)}{\partial x_i}^T = -\underbrace{\sum_{j \in \mathcal{N}(i)} w_{ij}(\|x_i - x_j\|)(x_i - x_j)}$$

$\mathcal{E}_{ij}$: pair-wise *edge-tension energy*

$\mathcal{E}_{ij}(\|x_i - x_j\|) = 0$ if $\|x_i - x_j\| = d_{ij}$

weighted consensus protocol (w_{ij} depends on $||x_i - x_j||$)





Motivation

- Assume some agents (leaders) can be arbitrarily controlled
  - Remaining agents (followers) obey the original interaction rule
  - Leaders' motion is considered as the inputs to the network

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ \\ x_N \end{bmatrix} \rightarrow \quad x_f \in \mathbb{R}^{N_f d}, \quad x_\ell \in \mathbb{R}^{N_\ell d}$$

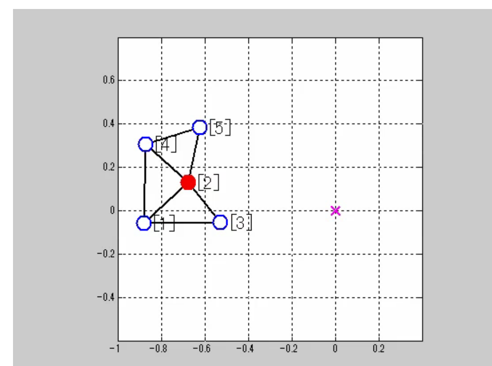
$$\dot{x}_f(t) = -\frac{\partial \mathcal{E}(x_f, x_\ell)}{\partial x_f}^T$$

$$\dot{x}_\ell(t) = u(t)$$

$$\mathcal{E}(x(t)) = \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N} \mathcal{E}_{ij}(\|x_i - x_j\|)$$

$$\mathcal{E}_{ij} = 0 \text{ if } (\mathrm{i},\mathrm{j}) \notin \mathbf{E}$$

(N_f followers and N_ℓ leaders)



How the network can be adaptively changed to maximize the effectiveness of leader's inputs?

How can we measure the impact of leader's input to the network?

Evaluate Leader's Influence on the Network

- Reachability/controllability [Rahmani,Mesbahi&Egerstedt 2009]
  - Global point-to point property (arbitrary configurations)
  - Long-term index
- Instantaneous network response
  - Local property (depends on a particular configuration)
  - Short-term index

The instantaneous index can be more useful under a dynamic situation (e.g., change the topology adaptively to maximize the leader's effect)

Manipulability index of leader-follower networks

[Kawashima&Egerstedt, CDC2011]

Responsiveness

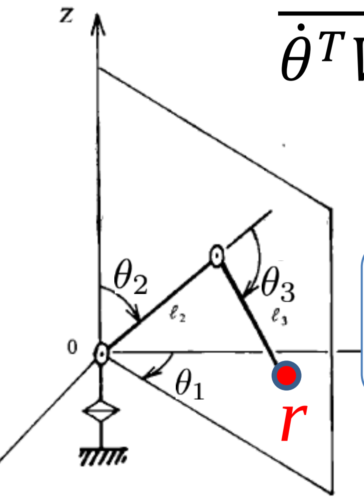
Robot-arm manipulability

[Yoshikawa 1985, Bicchi, et al. 2000]

$$\frac{\dot{r}^T W_r \dot{r}}{\dot{\theta}^T W_\theta \dot{\theta}}$$

end-effector velocity

angular velocity of joints



r : states of end-effector
θ : joint angles
$W_r, W_\theta \succ 0$: weight matrices

Kinematic relation

$$r = f(\theta), \quad \dot{r} = \left.\frac{\partial f}{\partial \theta}\right|_\theta \dot{\theta}$$

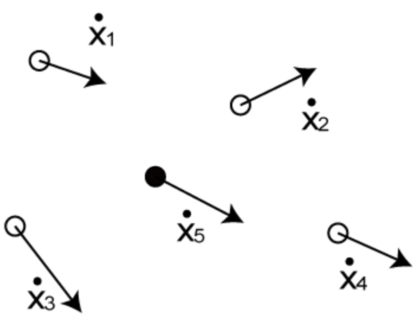
Velocity of end-effector is directly connected with the angular v

Leader-follower manipulability

$$m = \frac{\dot{x}_f{}^T Q_f \dot{x}_f}{\dot{x}_\ell{}^T Q_\ell \dot{x}_\ell}$$

followers' vel.

leaders' vel.



Dynamics of agents

$$\dot{x}_\ell(t) = u(t)\text{: given}$$

$$\dot{x}_f(t) = -\frac{\partial \mathcal{E}(x_f, x_\ell)}{\partial x_f}^T$$

We need integral action w.r.t. time

Can we get more direct connection between $\dot{x}_\ell(t)$ and $\dot{x}_f(t)$?

Approximation of the Dynamics

<u>Def: Rigid-link approximation</u> [Kawashima&Egerstedt CDC2011]

– Followers are fast enough to perfectly maintain the desired distance d_{ij} $\forall (i,j) \in \mathbf{E}$ $\forall t$ ($\mathbf{E}$: edge set)

- Under this approximation,

$$\frac{d}{dt}||x_i - x_j||^2 = 0 \qquad \forall (i,j) \in \mathbf{E}$$

$$\therefore \quad (x_i - x_j)^T(\dot{x}_i - \dot{x}_j) = 0 \qquad \forall (i,j) \in \mathbf{E}$$

Constraint on each connected agent pair

- Rewrite with the rigidity matrix [B.Roth 1981]

$$R(x) \in \mathbb{R}^{|\mathsf{E}| \times Nd}$$

$$R(x)\dot{x} = \begin{bmatrix} & & \cdots & & \\ & & \cdots & & \\ 0^T & (x_i - x_j)^T & 0^T & -(x_i - x_j)^T & 0^T \\ & & \cdots & & \\ & & \cdots & & \end{bmatrix} \begin{bmatrix} \vdots \\ \dot{x}_i \\ \vdots \\ \dot{x}_j \\ \vdots \end{bmatrix} = 0$$

Manipulability under the Rigid-link Approximation

- Theorem [Kawashima&Egerstedt CDC2011]

Given the rigidity matrix $R(x) \in \mathbb{R}^{|\mathsf{E}| \times Nd}$

$$R(x) = \Big[\underbrace{R_f(x)}_{N_f d}\Big|\underbrace{R_\ell(x)}_{N_\ell d}\Big]$$

$$\dot{x}_f(t) = -\frac{\partial \mathcal{E}(x)}{\partial x_f}^T \quad \sim \quad \dot{x}_f(t) = \boxed{-R_f^\dagger R_\ell}\,\dot{x}_\ell(t)$$

$$J(x, E) \triangleq -R_f^\dagger R_\ell$$

  - Provides us the direct connection between $\dot{x}_\ell(t)$ and $\dot{x}_f(t)$

Manipulability of leader-follower networks

$$m = \frac{\dot{x}_f{}^T Q_f \dot{x}_f}{\dot{x}_\ell{}^T Q_\ell \dot{x}_\ell}$$

$\sim$

Approximate manipulability

$$\hat{m}(\dot{x}_\ell, x, \mathbf{E}) = \frac{\dot{x}_\ell{}^T J^T Q_f J \dot{x}_\ell}{\dot{x}_\ell{}^T Q_\ell \dot{x}_\ell}$$

$\hat{m}$ can evaluate the instantaneous effectiveness leader's input $\dot{x}_\ell$ by considering current config.(x) and network topology ($\mathbf{E}$).

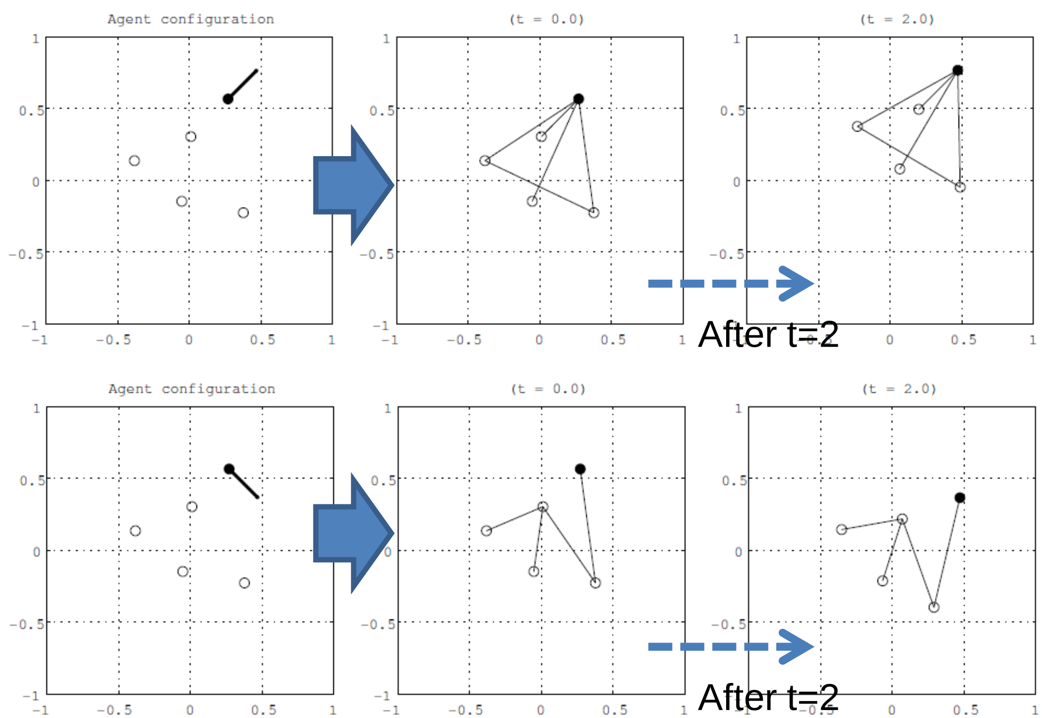
Application Example: Effective Topology

- Given leader's input direction, what is the most effective network topology in terms of maximizing the manipulability?

Topology optimization

$$\underset{\mathbf{E}}{\operatorname{argmax}} \quad \hat{m}(\dot{x}_\ell, x, \mathbf{E}) = \frac{\dot{x}_\ell^{\,T} J^T Q_f J \dot{x}_\ell}{\dot{x}_\ell^{\,T} Q_\ell \dot{x}_\ell}$$

(The number of edges, $|\mathbf{E}|$, is constrained)



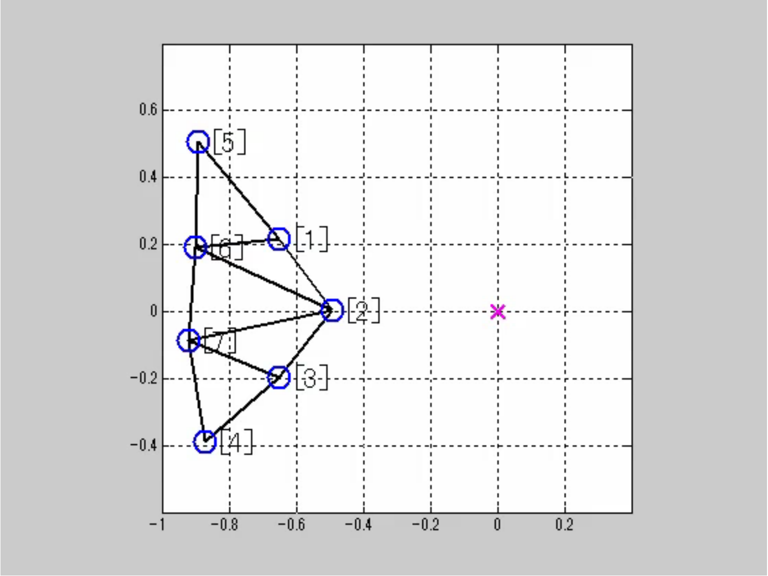
Application Example: Online Leader Selection

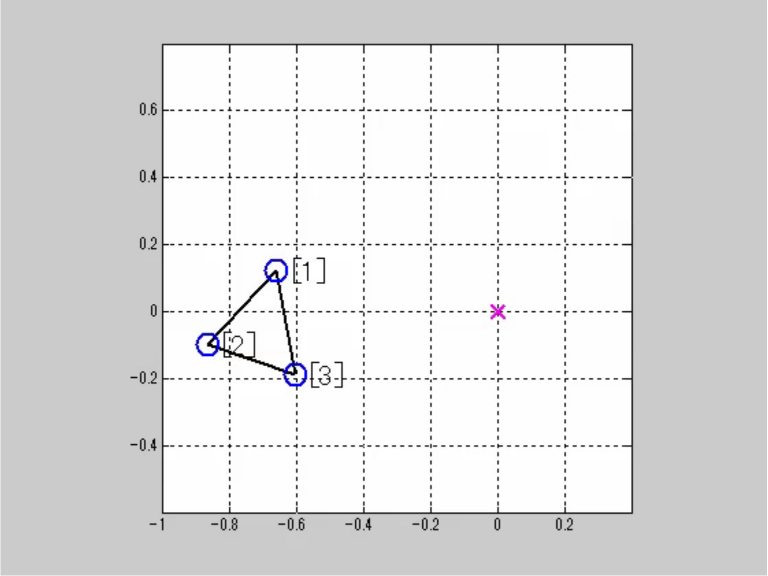
[Kawashima&Egerstedt, ACC2012]

- To move the network toward a target point, which is the most effective leader

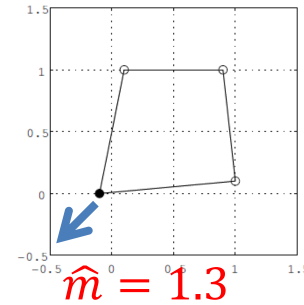
Online leader selection

$$i_\ell(t) = \underset{\color{red}{i}}{\operatorname{argmax}}\, \hat{m}_e(\color{red}{i}\color{black}, x(t), \mathbf{E})$$



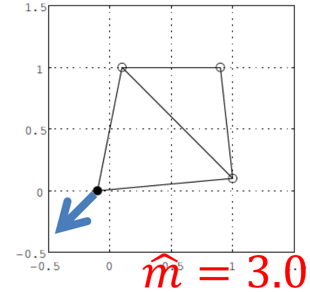


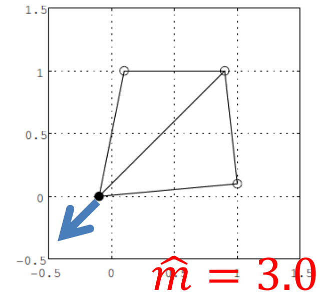
Limitation of Manipulability



☹1. Manipulability cannot compare "rigid formations" with the same agent configurations.

☹2. Manipulability cannot deal with "edge weights" (i.e., gains of the pair-wised edge-tension energies).





Why?

In the rigid-link approximation, we only considered <u>the convergence point</u> of the followers when the leader is moved.

How can we overcome these limitations?

Stiffness and Rigidity Indices

[Zhu&Hu, CDC09, CDC11]

- Stiffness matrix is defined based on the spring-mass analogy, and it coincides with the Hessian of the edge-tension energy $\mathcal{E}$, in the formation-control context:

$$\delta\dot{x}(t) = -H\delta x(t),\ \ \delta x(0) = \delta x_0 \quad \text{where} \quad H \triangleq \left.\frac{\partial^2 \mathcal{E}}{\partial x^2}\right|_{x=x^*}$$

- *Rigidity indices* are defined by the eigenvalues of the Hessian
  - Worst-case rigidity index (WRI): smallest eigenvalue of H: $\lambda_r(H), r = \text{rank}(H) \quad (\lambda_k \geq \lambda_{k+1})$

It dictates how fast the formation can be recovered

Complementary with the manipulability

Responsiveness

- Unifies the notions of *stiffness* and *manipulability*
  - Analyze the convergence process of the followers

$$\nu(t, x, G, \delta x_\ell) = \frac{\left\|\delta x_f(t)\right\|^2}{\|\delta x_\ell\|^2}$$

$\xrightarrow{t \longrightarrow \infty}$ Manipulability $\hat{m}$

t is often finite (leader cannot wait followers for infinite time)

Solve the following zero-state response with $\delta x_f(0) = 0$:

$$\delta \dot{x}_f(t) = -H_{ff}\delta x_f(t) - H_{f\ell}\delta x_\ell, \quad \text{where } H_{ff} \triangleq \left.\frac{\partial^2 \mathcal{E}}{\partial x_f^2}\right|_{x^*}, H_{f\ell} \triangleq \left.\frac{\partial^2 \mathcal{E}}{\partial x_f \partial x_\ell}\right|_{x^*}$$

using the fact $H_{f\ell} = -H_{ff}\left(\mathbf{1}_{N_f} \otimes I_d\right)$ and the decomposition $H_{ff} = V\Lambda V^T$

$$\delta x_f(t) = V\mathbf{Diag}\left(1 - e^{-\lambda_k t}\right)V^T\left(\mathbf{1}_{N_f} \otimes I_d\right)\delta x_\ell$$

$\Lambda = \mathbf{Diag}(\lambda_1, \dots, \lambda_r)$: non-zero eigenvalues

$V = [v_1, \dots, v_r]$: eigen vec.

Responsiveness

$$\delta x_f(t) = \overbrace{V\mathbf{Diag}(1 - e^{-\lambda_k t})V^T \left(\mathbf{1}_{N_f} \otimes I_d\right)}^{J(t)} \delta x_\ell$$

$$\nu(t, \delta x_\ell, x, \mathbf{E}) = \frac{\left\|\delta x_f(t)\right\|^2}{\|\delta x_\ell\|^2} = \frac{\delta x_\ell^T J(t)^T J(t) \delta x_\ell}{\delta x_\ell^T \delta x_\ell}$$

$$= \sum_{k=1}^{r} \underbrace{\left(1 - e^{-\lambda_k t}\right)^2}_{\text{Stiffness (rate of convergence)}} \underbrace{\left(v_k^T \left(\mathbf{1}_{N_f} \otimes I_d\right) \widehat{\delta x_\ell}\right)^2}_{\text{Manipuability (convergence point)}}$$

$$H_{ff} = \left.\frac{\partial^2 \mathcal{E}}{\partial x_f^2}\right|_{x^*} = V \Lambda V^T$$

Non-zero eigs.

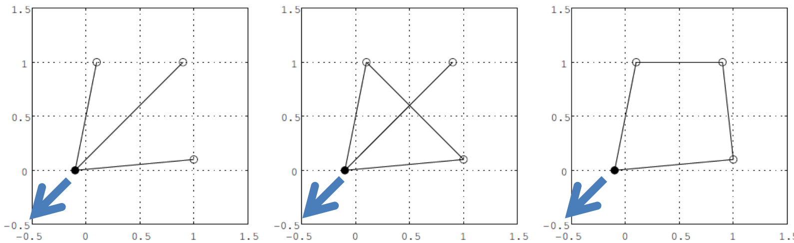
$$\Lambda = \mathbf{Diag}(\lambda_1, \dots, \lambda_r)$$

$$V = [v_1, \dots, v_r]$$

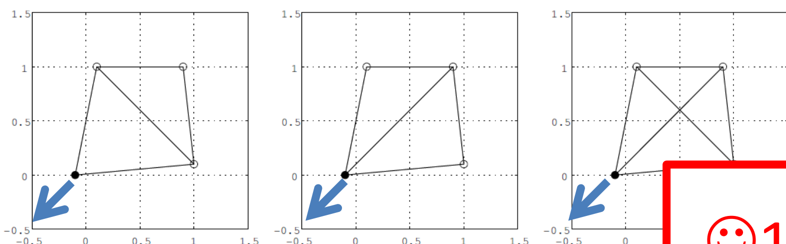
- Responsiveness takes into account the rate of convergence and the convergence point simultaneously.
  - How fast followers respond? (evaluated by given time t)
  - How much followers finally respond? (evaluated by given input, i.e., leader's motion, δx_ℓ)

Comparison of Responsiveness

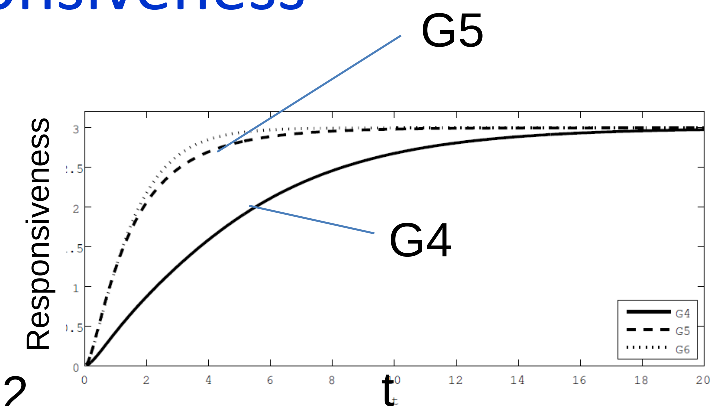
$$\nu(t,\delta x_\ell,x,\mathbf{E}) = \sum_{k=1}^{r}\left(1-e^{-\lambda_k t}\right)^2\left(v_k^T\left(\mathbf{1}_{N_f}\otimes I_d\right)\ \widehat{\delta x_\ell}\right)^2$$



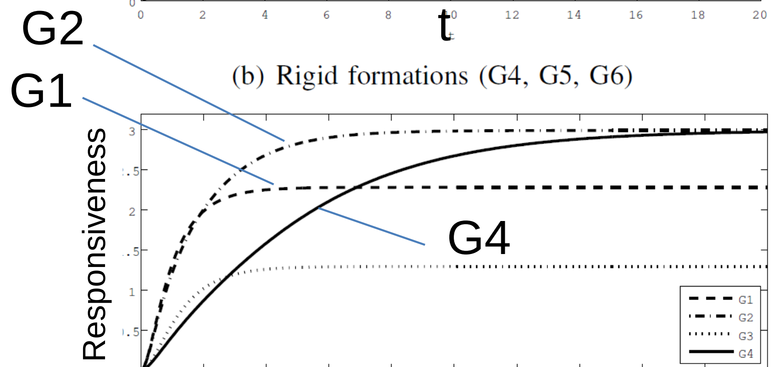
(a) G1 (b) G2 (c) G3



(d) G4 (e) G5 (f)



(b) Rigid formations (G4, G5, G6)



☺1. Responsiveness can be used to compare rigid formations

Formation	Is rigid?	r	Resp. $\nu(t=2$				
G1	No	3	1.985	[illegible]	[illegible]	[illegible]	[illegible]
G2	No	4	2.0092	2.9838	2.9912	(0.6257)	(1.2288)
G3	No	4	1.0212	1.2960	1.2961	(0.9581)	(1.2457)
G4	Yes	5	0.8717	2.6740	2.9953	0.2710	0.8025
G5	Yes	5	2.0583	2.9794	2.9953	0.4771	1.0279
G6	Yes	5	2.1791	2.9946	2.9953	0.8918	1.5220

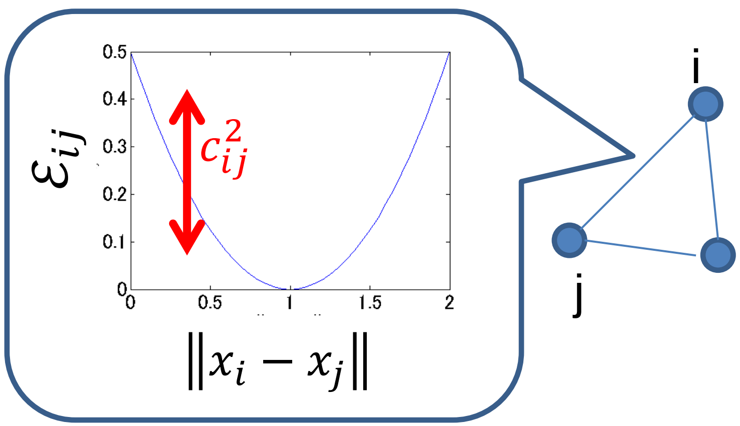
Optimal Edge Weights (Link Resource Allocation)

- Link weights can be viewed as the amount of resource.
- Given limited amount of total resource, find the optimal link resource allocation in terms of maximizing the convergence rate of the responsiveness $\lambda_r(H_{ff})$.

$$\underset{c_{ij}}{\text{maximize}}\ \lambda_r(H_{ff})$$

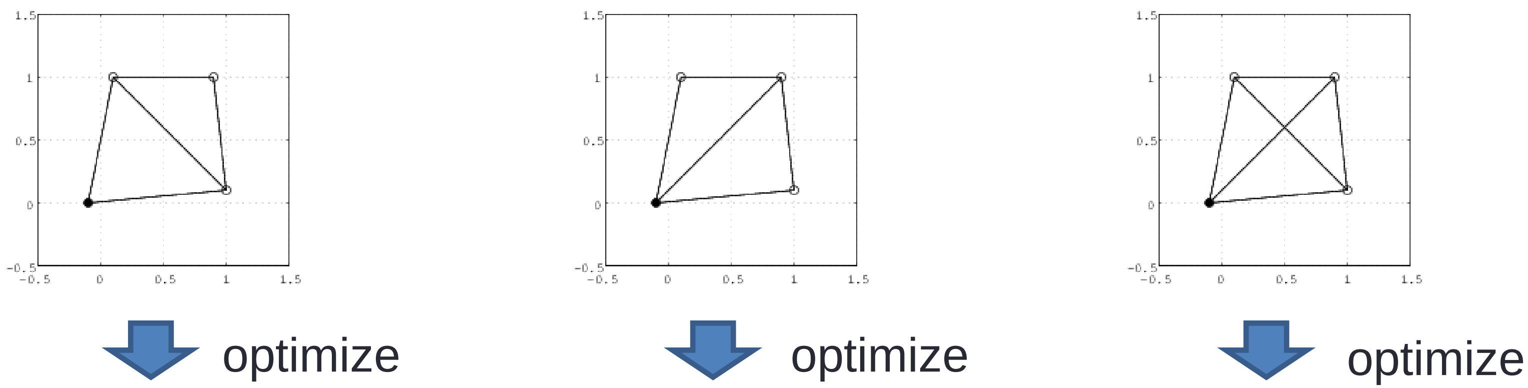
$$\text{subject to} \sum_{\{i,j\}\in \mathrm{E}} c_{ij}^2 \leq C \text{ and } c_{ij} = 0 \text{ for } \{i,j\} \notin \mathrm{E}$$

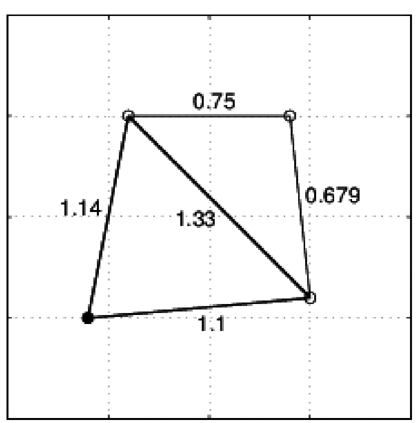
- r – rank of H_{ff} $\quad(\lambda_k \geq \lambda_{k+1})$
- C – total amount of resource
- c_{ij} – weight for link $\{i,j\}$

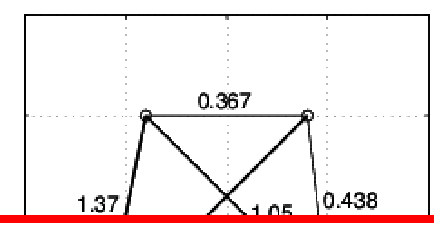


Optimal Edge Weights (Link Resource Allocation)

Numerical Results







☺2. Responsiveness can deal with edge weights

Conclusion

- We proposed the *responsiveness* of leader-follower networks, which unifies the previously defined notions of *stiffness* and *manipulability*.
- It measures the impact of leader's input in terms of
  - how fast / how large (finally) followers respond.

$$\nu(t, \delta x_\ell, x, \mathbf{G}) = \sum_{k=1}^{r} \underbrace{\left(1 - e^{-\lambda_k t}\right)^2}_{\text{Stiffness}} \underbrace{\left(v_k^T \left(\mathbf{1}_{N_f} \otimes I_d\right) \widehat{\delta x_\ell}\right)^2}_{\text{Manipulability}}$$

- Future work
  - Application to *human-swarm interaction* (e.g., networked robots adaptively changes gains and topologies to maximize human inputs injected through the leader).

Thank you for your attention!