Optimal Switching Control of a Step-Down DC-DC Converter

H. Kawashima, Y. Wardi, D. Taylor, and M. Egerstedt

***Abstract*— This paper applies a general computational technique for optimal control of switched-mode hybrid systems, recently developed by the authors, to voltage-regulation problems in a step-down DC-DC converter. Unlike existing techniques that are based on model-predictive control and a specific algebraic structure of the problem, the algorithms presented here are based on gradient descent with Armijo step size, and consequently can incorporate time-dependent state constraints in a natural way. The approach proposed in this paper is complementary to the extant MPC-based techniques, and it appears to compare favorably with some of the established works. Two problems are being addressed: one concerns pulse-width modulation and computes the optimal duty ratio, and the other computes an optimal switching schedule without a fixed cycle time.**

I. Introduction

The problem of hybrid optimal control has been the focus of extensive research in the past decade. One of the major application areas motivating this problem is in power electronics, and more specifically in switching circuits, where it is desirable to regulate or control a state variable, such as current or voltage, by a switching schedule. In a typical problem the objective of the switching control is to minimize the deviation of a state variable from a given reference value while satisfying pointwise inequality constraints on other state variables. The purpose of this paper is to provide a test case of a general algorithmic approach, recently developed by the authors, for voltage regulation in a DC-DC buck converter. Two problems will be considered: in the first, the schedule will be defined via the duty ratios of the switch for a given switching frequency, and in the second, an optimal, acyclic schedule will be computed without imposing a switching frequency. A particular feature of our approach is that it incorporates state constraints over a continuum of time-points in a natural way.

Such switching control problems have been extensively considered by Morari et al.; see [9], [1], [2] and references therein. Their approach is based on model predictive control over a finite horizon, and piecewise-affine interpolations of measurements taken at sample data points. The optimal switching times are computed by mixed integer linear or quadratic programming. An alternative algorithmic approach has been developed by DeCarlo et al. in [5], [11], [12] (also see references therein). It is based on relaxations of the optimal mode-scheduling problem, solving the relaxed problem by nonlinear model-predictive control techniques, and computing the real-time switching control laws.

The method proposed in this paper is based on gradient-descent techniques rather than on linear functional approximations, and it computes directly in the space of mode-schedules without resorting to relaxations. It has been developed in a general setting of nonlinear, autonomous, optimal mode-switching problems [16], and tested on a simple academic example. In this paper we tailor it to specific problems arising in optimal voltage regulation in a DC-DC converter and demonstrate its computational efficacy. In particular, we will point out its handling of state-variable constraints without having to resort to sampling, thereby avoiding the potential problem of state ripples and associated constraint violations, which may arise in sample-based techniques. We will do some comparisons with a particular existing method, namely [2] since it served to motivate the results described in this paper, but we point out that broad comparisons with the existing methods is premature at this time since our technique is still in its infancy. Our objective is but to introduce a new player to the field of optimal switching control.

The starting point of our investigation is Reference [2], where a Pulse-Width Modulation (PWM) problem in a DC-DC converter is considered. We solve the same problem and get a fast convergence of our algorithm. There is insufficient data for a detailed comparison of the two techniques, but both converge fast. However, it is noted that the algorithm in [2] does not always guarantee that pointwise state constraints are satisfied, and this may be due to the facts that it is based on sampling and a particular algebraic structure. In contrast, the algorithm described here addresses a continuum of constraints by integrating a penalty function within the structure of the cost function in a way that appears to limit the magnitude of state ripples, thereby avoiding constraint violations, while having little impact on the algorithm's computational workload.

The second problem we consider is to compute an optimal switching schedule that is not confined to the structure of PWM with fixed switching frequency. Instead, the switching schedule can be fairly general and independent of a particular control structure. This problem is more difficult than the PWM problem since its controlled variable consists not only of the switching times but also of the instantaneous switching rates, and hence, as we shall see, it has discrete as well as continuous components. The algorithm that we use has been developed for such problems, and its demonstrated efficacy suggests its eventual use in other application areas. We are cognizant of the fact that optimal switching-control problems

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in power electronics often are stated in the setting of PWM, but we believe that the more-general setting may become increasingly suitable to future applications involving leakage energy associated with switching. Energy-related costs are not being considered in this paper but will be addressed in a forthcoming publication.

Section II presents our optimization techniques in an abstract setting, while Section III applies them to the particular problems of controlling the switching schedules in DC-DC converters. Finally, Section IV concludes the paper and suggests directions for future research.

II. Optimization of Mode-Schedules in Switched-Mode Systems

In this section, we briefly review the optimal mode-scheduling algorithms that will be used in the sequel. A general autonomous switched-mode dynamical system can be described by the following equation,

$$\dot{x}(t) = f(x(t), v(t)), \tag{1}$$

where $x \in \mathbb{R}^n$ is the state variable, and the control $v(t)$ is confined to a finite set V. Suppose that the system evolves in a finite horizon $[0, T]$ for a given $T > 0$, and that the control function $v(t)$ has its values changed a finite number of times. We assume that the function $f(\cdot, v)$ is twice-continuously differentiable for every $v \in V$. The initial state $x_0 = x(0)$ is assumed to be given and fixed.

Given a continuously-differentiable function $L : \mathbb{R}^n \to \mathbb{R}$, consider the problem of minimizing the cost functional J, defined via

$$J := \int_0^T L(x(t))dt, \tag{2}$$

with respect to the control $v(t)$, $t \in [0, T]$. Given a control function $v(t)$, $t \in [0, T]$, let $v^1, ..., v^{N+1} \in V$ denote the successive control values $v(t)$ in $[0, T]$, and let τ_i denote the switching time between v^i and v^{i+1}, where $0 \leq \tau_1 \leq ... \leq \tau_N \leq T$. We denote the switching times by the vector notation $\bar{\tau} := (\tau_1, ..., \tau_N)^\top \in \mathbb{R}^N$; we further define $\tau_0 := 0$ and $\tau_{N+1} := T$. Using this notation, we rewrite the system equation (1) as

$$\dot{x}(t) = f_i(x(t)) \quad \forall t \in [\tau_{i-1}, \tau_i), \quad i = 1, ..., N+1, \tag{3}$$

where $f_i(x(t)) := f(x, v^i)$.

The successive values of $v \in V$, namely $v^1, \ldots, v^{N+1}$, represent the various modes of the system, and hence the problem of minimizing J subject to (3) can be viewed as an optimal mode-switching problem. We discern two kinds of problems: a *timing optimization problem*, and a *scheduling optimization problem*. The timing-optimization problem arises when the sequence of modes $\{v^1, \ldots, v^{N+1}\}$ is given, and it is desirable to compute the switching times between them, namely the vector $\bar{\tau} \in \mathbb{R}^N$. In the scheduling optimization problem, the controlled variable consists of the mode sequence $\{v^1, \ldots, v^{N+1}\}$ including the number of modes, $N + 1$, as well as the switching-time vector $\bar{\tau}$. Obviously the timing optimization problem is much easier than the scheduling optimization problem, since the former is a nonlinear programming problem with a continuous variable $\bar{\tau}$, whereas the latter is a mixed-integer problem involving the sequence of modes. The next two subsections describe algorithms for solving these problems, recently developed by the authors.

A. Algorithm for the Timing Problem

Given a mode-sequence, the problem is to minimize J, defined in (2), as a function of $\bar{\tau} := (\tau_1, \ldots, \tau_N)^\top$ subject to the constraints that $0 \leq \tau_1 \ldots \leq \tau_N \leq T$. We use a steepest-descent algorithm with Armijo step sizes, modified to project the gradient onto the feasible set to account for the inequality constraints $0 \leq \tau_1 \leq \ldots \leq \tau_N \leq T$. The partial derivatives $\frac{\partial J}{\partial \tau_i}$ are computed as follows. Denote by $F(x, t)$ the Right-Hand Side (RHS) of (3), so that $\dot{x} = F(x, t)$. Define the costate variable $p(t) \in \mathbb{R}^n$ by the following equation,

$$\dot{p}(t) = -\left(\frac{\partial F}{\partial x}(x, t)\right)^\top p(t) - \left(\frac{\partial L}{\partial x}(x)\right)^\top \tag{4}$$

with the boundary condition $p(T) = 0$. Then (see [8]), for all $i = 1, \ldots, N$,

$$\frac{\partial J}{\partial \tau_i}(\bar{\tau}) = p(\tau_i)^\top (f_i(x(\tau_i)) - f_{i+1}(x(\tau_i))). \tag{5}$$

The steepest-descent algorithm with Armijo step size has the following form.

Algorithm 1: Given: Constant parameters $\alpha \in (0, 1)$ and $\beta \in (0, 1)$.
Step 0: Choose an initial feasible point $\bar{\tau}_0$. Set $k = 0$.
Step 1: Compute $h_k := \nabla J(\bar{\tau}_k)$ using (5). If $||h_k|| = 0$, then exit; otherwise, continue.
Step 2: Compute $j(\bar{\tau}_k)$ defined by $j(\bar{\tau}_k) = \min\left\{j = 0, 1, ... : J(\bar{\tau}_k - \beta^j h_k) - J(\bar{\tau}_k) \leq -\alpha\beta^j ||h_k||^2\right\}$, and set $\gamma(\bar{\tau}_k) := \beta^{j(\bar{\tau}_k)}$.
Step 3: Set $\bar{\tau}_{k+1} := \bar{\tau}_k - \gamma(\bar{\tau}_k)h_k$, set $k = k + 1$, and go to Step 1.

As mentioned earlier, compliance with the inequality constraints requires a modification of Algorithm 1 by projecting $-\nabla J(\bar{\tau}_k)$ onto the feasible set and possibly truncating h_k to ensure feasibility. For details, please see [3].

This algorithm is globally convergent and tends to yield large step sizes when far off a minimum; see [13] for extensive discussions including theory and applications.

B. Algorithm for the Scheduling Problem

Let Σ denote the space of mode-sequences corresponding to control functions $v : [0, T] \to V$ having a finite (but not necessarily bounded) number of switching points. The problem considered here is to minimize J, defined by (2), as a function of the mode-schedules $\sigma \in \Sigma$. The space Σ is infinite dimensional and incomplete, and this renders challenging the task of developing provably-convergent optimization algorithms.

A number of algorithms for this general scheduling problem have emerged, including geometric techniques [6], [14], [15], relaxation methods [5], [7], and steepest-descent algorithms [4], [10]. The latter algorithms insert a single mode to a given schedule and solve the resulting timing optimization problem. Recently, a new descent-based technique has been proposed, which iterates directly in the schedule space and avoids the need for solving timing optimization problems [16]. It can swap any number of modes on large time-sets at a given iteration, and it uses the Armijo step size to compute the Lebesgue measure of the time-set where modes are to be changed. It appears to yield large descents from schedules that are far-off from optimum points, and hence we expect it to converge fast towards a minimum. It is this algorithm that we try in this paper on the switching-circuit regulation problem.

To present the details of this algorithm, let us define the concept of the *insertion gradient* as follows [4]. Given a mode-schedule $\sigma \in \Sigma$, a time $s \in [0,T)$, and a point $w \in V$, consider inserting the mode associated with w to σ at time s for a duration of $\lambda > 0$ seconds, and consider the resulting cost functional, J, defined by (2), as a function of $\lambda > 0$. The insertion gradient of w at (σ, s), denoted by $D_{\sigma,s,w}$, is defined as $D_{\sigma,s,w} = \frac{dJ}{d\lambda^+}(0)$. We recognize in this a needle variation, and consequently we have the following expression,

$$D_{\sigma,s,w} := \frac{dJ}{d\lambda^+}(0) = p(s)^\top \big(f(x(s),w) - f(x(s),v(s))\big), \tag{6}$$

where the costate p is defined by (4). Now a word must be said about w in Equation (6). If $w = v(s)$ then $D_{\sigma,s,w} = 0$ since no change is made to the schedule. On the other hand, if $w \neq v(s)$ then $D_{\sigma,s,w}$ may be negative, positive, or 0. If $D_{\sigma,s,w} < 0$ then inserting to σ the mode associated with w for a brief amount of time starting at s would result in a decrease in J. We seek such an insertion yielding a descent as large as possible, and therefore we define, for a given $\sigma \in \Sigma$,

$$D_\sigma \;=\; \min_{s \in [0,T]} \min_{w \in V} (D_{\sigma,s,w}).$$

Then the necessary optimality condition equivalent to stationarity is that $D_\sigma = 0$.

To simplify the presentation of the algorithm, suppose that there are only two modes, namely the set V contains two elements. Thus, a mode-insertion to a given schedule means that, at the insertion time $s \in [0,T]$, the current mode is flipped, i.e., replaced by its complementary mode. The main idea behind the algorithm is to flip the modes at times in a set which is as large as possible in order to guarantee a large descent in J. This set is computed via an Armijo procedure, and it may be disconnected. The set of points $s \in [0,T]$ where $D_{\sigma,s} < 0$ is unsuitable for this purpose, so we search for a subset thereof where $D_{\sigma,s}$ is "more negative". Fix $\eta > 0$, and define $S_{\sigma,\eta} := \{s \in [0,T] : D_{\sigma,s} \leq \eta D_\sigma\}$. The algorithm looks for a subset of $S_{\sigma,\eta}$ in the following way. Let $\mu(set)$ denote the Lebesgue measure of the set set, and let $S : [0, \mu(S_{\sigma,\eta})] \to 2^{S_{\sigma,\eta}}$ be a mapping such that, $\forall \lambda \in [0, \mu(S_{\sigma,\eta})]$, $S(\lambda)$ is the finite union of intervals, and $\mu(S(\lambda)) = \lambda$. Such a mapping is not unique, but a good choice for it is to have $S(\lambda)$ be the leftmost subset of $S_{\sigma,\eta}$, S, such that $\mu(S) = \lambda$. Furthermore, define $\sigma(\lambda)$ to be the schedule obtained from σ by flipping the mode at every time-point $s \in S(\lambda)$. The algorithm has the following form (see [16]).

Algorithm 2: Given: Constant parameters $\eta \in (0,1)$, $\alpha \in (0,\eta)$, and $\beta \in (0,1)$.
Step 0: Choose an initial schedule $\sigma_0 \in \Sigma$. Set $k = 0$.
Step 1: Compute D_{σ_k}. If $D_{\sigma_k} = 0$, then exit; otherwise, continue.
Step 2: Compute $S_{\sigma,\eta}$.
Step 3: For $\lambda^j := \beta^j \mu(S_{\sigma,\eta})$, compute $j(\sigma_k)$ defined by

$$j(\sigma_k) = \min\left\{ j = 0,1,\ldots : J(\sigma_k(\lambda^j)) - J(\sigma_k) \leq \alpha \lambda^j D_{\sigma_k} \right\}.$$

Define $\lambda_k := \lambda^{j(\sigma_k)}$.
Step 4: Set $\sigma_{k+1} := \sigma_k(\lambda_k)$, set $k = k+1$, and go to Step 1.

This algorithm is suitable for unconstrained problems, and a modification of it to the case involving continuum of constraints will be described in the next section.

C. Penalty Function for State Inequality Constraints

Consider the problem of minimizing J as defined by Equations (1) and (2) (either the timing optimization problem or the scheduling optimization problem) subject to the additional requirement of satisfying a continuum of state constraints. These constraints are expressed via inequalities of the form $G(x(t)) \leq 0$ for every $t \in [0,T]$, where $G : \mathbb{R}^n \to \mathbb{R}$ is a continuously-differentiable function. The idea we pursue is to use a penalty function, and what comes to mind is an application of a scaled ramp function on $G(x(t))$. Thus, denoting the ramp function by $\mathcal{R}(z)$ for $z \in \mathbb{R}$, namely $\mathcal{R}(z) = z$ for $z \geq 0$ while $\mathcal{R}(z) = 0$ for $z < 0$, the penalty term is $c\mathcal{R}(G(x(t)))$ $\forall t \in [0,T]$ for a suitably-large $c > 0$. Furthermore, we can integrate these terms over $t \in [0,T]$ together with the cost function $L(x(t))$ in order to avoid having to handle a continuum of penalty terms, and the result is a minimization of the function

$$\int_0^T \Big(L(x(t)) + c\mathcal{R}\big(G(x(t))\big)\Big) dt \tag{7}$$

over the switching times $\bar{\tau}$ or the schedules $\sigma \in \Sigma$, depending on whether the timing optimization problem or the scheduling optimization problem is under consideration. Since $x(t)$, $G(x)$, and $\mathcal{R}(z)$ are continuous, this ensures that $\forall \epsilon > 0$, the Lebesgue measure of the subset of $[0,T]$ where $G(x(t)) > \epsilon$ converges to 0 as $c \to \infty$.

The term in (7) almost fits the framework defined by Equations (1) and (2), except for the fact that the function $L(x)$ in (2) is assumed to be continuously differentiable while the function $\mathcal{R}\big(G(x)\big)$ is not differentiable. Consequently, we approximate the ramp function $\mathcal{R}(z)$ by the parameterized set of functions $\mathcal{R}_a(z)$, $a > 0$, defined by

$$\mathcal{R}_a(z) \;=\; \frac{1}{a} \ln(e^{az} + 1). \tag{8}$$

$\mathcal{R}_a(z)$ is the integral of the sigmoid function $r_a(z) := \frac{1}{1+e^{-az}}$ which is known to approximate the step function, and hence $\mathcal{R}_a(z)$ approximate $\mathcal{R}(z)$ as $a \to \infty$ in the following

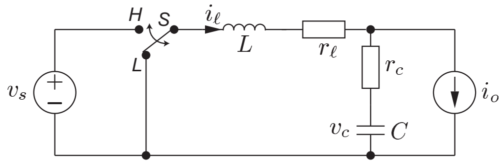

Fig. 1. Synchronous step-down DC-DC converter.

sense: For every $z \in \mathbb{R}$, $\lim_{a\to\infty} \mathcal{R}_a(z) = \mathcal{R}(z)$.[1] $\mathcal{R}_a(z)$ is continuously differentiable in z, and hence we use it to replace $\mathcal{R}(z)$ in minimizing the term in (7). Both parameters $c > 0$ and $a > 0$ can be fixed at large values or increased dynamically, but in any case, for given values of them, we minimize the function $J_{a,c}$, defined via

$$J_{a,c} \;=\; \int_0^T \Big(L(x(t)) + c\mathcal{R}_a\big(G(x(t))\big)\Big)dt \qquad (9)$$

in terms of the relevant timing or scheduling parameters.

III. Optimal Switching-Control of a DC-DC Converter

This section considers the same circuit, with the same parameter values, that was analyzed in [2]. We first solve the same optimal mode-switching problem that was considered in [2], and then we formulate and solve a mode-scheduling optimization problem.

The circuit shown in Figure 1 depicts a two-port DC-DC power converter, consisting of an ideal switch and a non-ideal LC filter, an ideal voltage source to supply the power that flows into the input port, and an ideal current source that absorbs the power that flows from the output port. Application of physical laws yields the dynamic model

$$\frac{d}{dt}\begin{bmatrix} v_c \\ i_\ell \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{C} \\ -\frac{1}{L} & -\frac{r_c+r_\ell}{L} \end{bmatrix}\begin{bmatrix} v_c \\ i_\ell \end{bmatrix} + \begin{bmatrix} -\frac{1}{C} \\ \frac{r_c}{L} \end{bmatrix} i_o + \begin{bmatrix} 0 \\ \frac{1}{L} \end{bmatrix} v_s v, \quad v \in \{0,1\} \qquad (10)$$

where v_c is the voltage across the ideal capacitance C, i_ℓ is the current through the ideal inductance L, r_c and r_ℓ represent the parasitic resistances in series with the ideal energy storage elements, v_s is the possibly time-varying voltage across the input source, i_o is the possibly time-varying current through the output load, and v is the control signal that determines the state of the switch.[2]

Following [2] we set the following values to the various circuit elements: $C = 70/2\pi$ farad, $L = 3/2\pi$ henry, $r_c = 0.005$ ohm, and $r_\ell = 0.05$ ohm. The rationale behind these values, given in [2], is they represent a scaled system of units.

[1]The sigmoid function, also called the logistic function, is used in pattern classification and other application areas requiring the mapping of continuous data into a binary set.

[2]Note the use of the nonstandard notation v for the control signal. We do this in order to comply with an emerging notation in the literature on switched-mode optimization, where v denotes the switching input signal while u denotes a continuous input control.

Observe that the numerical values of $v(t)$ in Equation (10) reflect on the position of the switch: If $v(t) = 1$ then the last term on the RHS of (10) is present, which corresponds to the state equation when the switch is in the H position. On the other hand, if $v(t) = 0$ then the last term in (10) is absent, corresponding to the state equation when the switch is in the L position.

The forthcoming optimization problems will be cast in the setting of receding horizons over brief intervals. We assume that $v_c(t)$ and $i_o(t)$ are measured at the start of each horizon interval and their obtained values are maintained throughout the interval, and therefore Equation (10) falls within the framework defined by (1).

A. Switching Time Optimization for PWM Systems

Consider a cyclical switching policy with a fixed switching period (cycle time) of T_s seconds. Since T_s is a constant, we count time in terms of numbers of cycles. Suppose that at each cycle the switch starts at the H state and then transitions to the L state; see Figure 1. Let the kth cycle consist of the time-interval $[k, k+1)$, $k = 1, 2, \ldots$, and let d_k denote the duty ratio of that cycle. Then, the switch is in the H state during the interval $[k, k+d_k)$, and it is in the L state during the rest of the cycle, namely in the interval $[k+d_k, k+1)$.

In the problem considered here (and in [2]) it is desirable to regulate the capacitor voltage $v_c(t)$ to a given reference value $v_{c,ref}$ by the duty ratios, while constraining the inductor current $i_\ell(t)$ to below a given threshold value, $i_{\ell,max}$ at all time t. In the context of PWM we define a receding horizon of M cycles for a given $M \geq 1$, and consider the following optimal control problem at the starting time of each cycle, k: Measure the input voltage v_s and i_o at time k, namely $v_s(k)$ and $i_o(k)$; define $v(t)$ for all $t \in [k, k+M)$ as follows: for every $m = 0, \ldots, M-1$,

$$v(t) \;=\; \begin{cases} 1, & t \in [k+m, k+m+d_{k+m}) \\ 0, & t \in [k+m+d_{k+m}, k+m+1); \end{cases} \qquad (11)$$

and for the system associated with the start of the kth cycle, we set $v_s(t) = v_s(k)$ and $i_o(t) = i_o(k)$ $\forall t \in [k, k+M)$, i.e., we consider the sources to have the constant value that is measured at time k. The state equation for this system is given by (10) with the above piecewise-constant values of v_s and i_o, and the initial condition for (10) is $x(k) = (v_c(k), i_\ell(k))^\top$, whose components are also measured at time k.[3] Finally, the optimal control problem is to compute the switching vector $\bar{\tau} = (k+d_k, \ldots, k+M-1+d_{k+M-1})^\top$ that minimizes the performance term J defined by

$$J \;=\; \frac{1}{2}\int_k^{k+M} \big(v_c(t) - v_{c,ref}\big)^2 dt, \qquad (12)$$

subject to the constraints $i_\ell(t) \leq i_{\ell,max}$ for every $t \in [k, k+M]$. This problem is solved at the starting time of each cycle k for the prediction horizon of M cycles, and the problems at the various cycle times may be different from each other

[3]Since r_c is an internal resistance of the capacitor, the "measurement" of v_c could be derived from measurements of output voltage, output current and inductor current.

according to the measured values of $v_s(\cdot)$ and $i_o(\cdot)$ at the start of each cycle.

To test out the timing-optimization algorithm on this problem, we used $v_{c,ref} = 1$ and $i_{\ell,max} = 3$. We addressed the constraint by using the penalty function $\mathcal{R}_a(z)$ defined in (8) with $a = 50$, and solved the corresponding problem of minimizing the function $J_{a,c}$, defined by

$$J_{a,c} \;=\; \int_k^{k+M} \Big(\frac{1}{2}\big(v_c(t) - v_{c,ref}\big)^2 + \mathcal{R}_a\big(i_\ell(t) - i_{\ell,max}\big)\Big)dt, \qquad (13)$$

with the penalty term $c = 1.0$. As for the sources, we let the voltage source have the constant value $v_s(t) = 1.8$, while the current source had the value $i_o(t) = 1.0$ during the cycles 1-30 and 61-90 and $i_o(t) = 2.0$ during cycles 31-60, as shown in Figure 2(a). The prediction horizon was $M = 2$, the initial condition was $x(0) = (0, 0)^\top$, and the parameters α and β in Algorithm 1 were set to $\alpha = \beta = 0.5$.

Algorithm 1 was used to solve this PWM problem and the results are shown in Figure 2 (b)-(e). Part (b) of the figure shows the computed optimal duty ratios d_k as a function of the cycle-index k, and we discern spikes at cycles 31 and 61 that are due to the changes in the current source i_o. Part (c) shows the voltage v_c as a function of the cycles 1-90, and we discern a rise from an initial value of 0 to about the target value of $v_{c,ref} = 1$ in about 8 cycles. We also note small deviations from the desirable value right after cycles 30 and 60, and again this is due to the changes in $i_o(t)$; however, the algorithm recovers from them quickly. Part (d) highlights these deviations by zooming the graph of part (c) on the cycle-range of 20-70. Finally, part (e) shows the graph of the current, and it is evident that the constraint $i_\ell(t) \le 3$ is satisfied for every cycle.

Comparing these results to those obtained in [2] for the same problem, we note that in both cases v_c reached its desirable value quickly. However, in [2] the inequality constraint $i_\ell(t) \le i_{\ell,max}$ was violated during several cycles, and we ascribe this to the fact that the cycle times are the same as the switching period, and hence perhaps large enough to cause substantial deviations in the state variables. In contrast, our algorithm's ability to handle a continuum of state constraints is defined by a different kind of resolution, namely its sampling resolution used for numerical integration of the differential equations; this is much finer than the sampling resolution in [2] and hence it seems to reduce the magnitude of the state deviations.

B. Optimal Mode-Scheduling for Acyclic Systems

This subsection concerns an alternative problem to the PWM, where the switching regimes do not have cycles of a constant length. We assume the same system as in the previous section. Furthermore, we consider a given horizon of length T, and hence the number of switchings in the horizon is to be determined. The objective is to minimize a function like J defined in (2) subject to a continuum of inequality constraints on the state, and to this end we

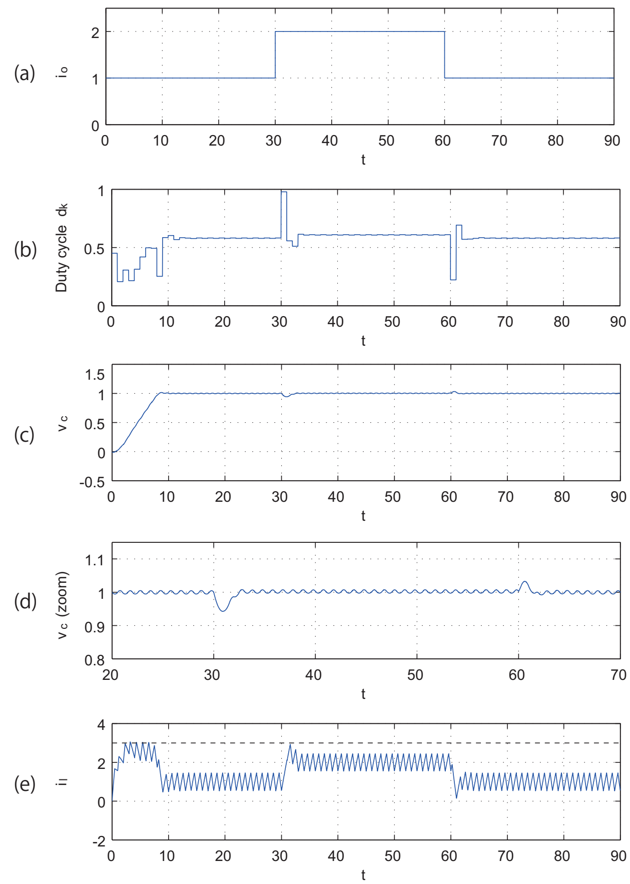

Fig. 2. Results of Algorithm 1 for the PWM problem

formulate the problem as a scheduling optimization problem and use Algorithm 2 for its solution. We point out that this problem was not considered in [2] and hence we do not have a basis for a close comparison.

The system is defined according to Equation (10), and the control v is associated with a mode-sequence $\sigma \in \Sigma$ as described in Section II. The cost functional $J(\sigma)$ is defined in a way similar to (12) except that the integral is taken over the horizon interval $[0, T]$, and thus,

$$J(\sigma) \;=\; \frac{1}{2}\int_0^T \big(v_c(t) - v_{c,ref}\big)^2 dt. \qquad (14)$$

We seek to minimize $J(\sigma)$ subject to the constraints $i_\ell(t) \le i_{\ell,max}$ for all $t \in [0, T]$. We apply the same penalty function approach as in the last subsection, and hence we minimize the function $J_{a,c}$ defined by

$$J_{a,c} \;=\; \int_0^T \Big(\frac{1}{2}\big(v_c(t) - v_{c,ref}\big)^2 + \mathcal{R}_a\big(i_\ell(t) - i_{\ell,max}\big)\Big)dt. \qquad (15)$$

We set $T = 20$, $\eta = 0.9$ in Algorithm 2, and $v_s(t) = 1.8$ and $i_o(t) = 1.0 \; \forall t \in [0, T]$; all other parameters have the same values as for the problem described in the previous subsection.

The initial value of the control was set to consist of a

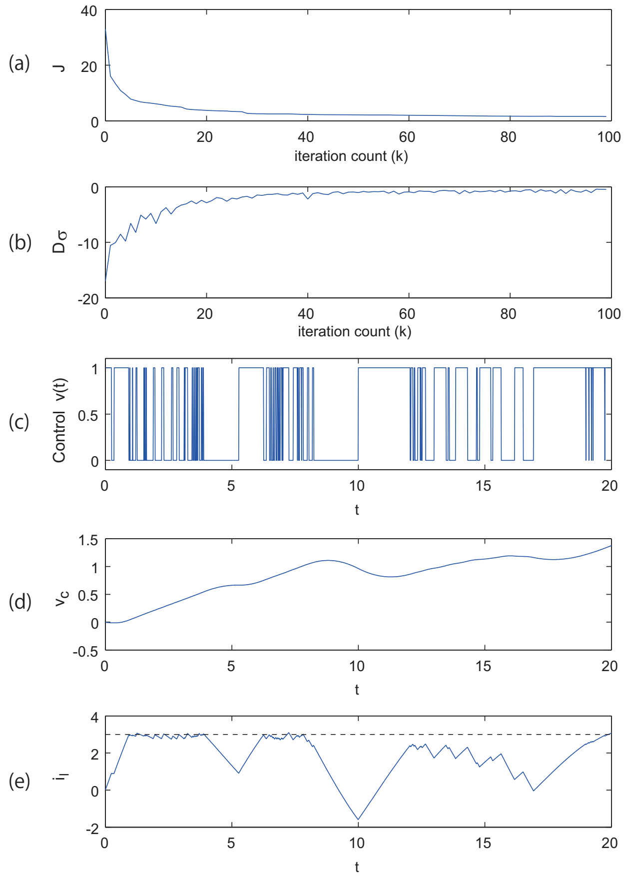

Fig. 3. Result of Algorithm 2 for the scheduling optimization problem

single switching from L to H at the mid-point of the horizon interval, namely, $v(t) = 0$ for $t \in [0, \frac{T}{2})$ and $v(t) = 1$ for $t \in [\frac{T}{2}, T]$. The algorithm was run for 100 iterations, and the results are shown in Figure 3. Part (a) depicts the value of the performance functional $J(\sigma_k)$ as a function of the iteration count k, and we clearly see a decrease from the initial value of 33.0 to 1.6, most of it in the first 10 iterations. Part (b) shows the graph of D_σ for the surrogate cost functional $J_{a,c}$, and we discern a convergence to 0, which suggests that the computed schedules converge towards a stationary point. For the last iteration, the graph of the control $v(t)$, $t \in [0, T]$, is shown in part (c), where we note that the switching becomes increasingly rapid in certain regions, suggesting a sliding mode at the limit. Part (d) shows the graph of the associated capacitor voltage $v_c(t)$, and we see that it hovers around the desirable value of 1 after about 8 iterations. Finally, part (e) depicts the graph of the corresponding inductor current $i_\ell(t)$, and we see that it violates the constraints only by minor, and barely discernible amounts.

IV. Conclusions

This paper considers the problem of optimal mode-scheduling in the specific context of switching circuits, and it tests two algorithms, recently developed by the authors, for its solution. The general problem addressed in the paper is to regulate the voltage in a DC-DC converter circuit by determining a suitable switching regime while observing upper-bound constraints on the current. Two specific problems are considered: the first is to optimize the duty ratio in a PWM system with fixed cycle times, and the second is to determine an optimal switching schedule without regard to a fixed cycle time. In both cases the algorithms exhibited fast convergence, and were shown to handle the constraints in an efficient manner.

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